ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH A SINGLE EXCEPTIONAL INFINITESIMAL TRANSFORMATION.

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Introduction.—Let $X_1, \cdots, X_r, X_{r+1}$ be the symbols of the infinitesimal transformations of a finite continuous group G with $r+1$ essential parameters, in which case

$$(1) \qquad (X_i, X_j) = \sum_{k=1}^{r+1} c_{ijk} X_k{}^* \quad (i, j = 1, 2, \cdots, r+1).$$

The symbols of the infinitesimal transformations of the group Γ adjoint to G are then

$$(2) \qquad D_i = \sum_{j=1}^{r+1} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \dagger \qquad (i = 1, 2, \cdots, r+1);$$

and we have

$$(3) \qquad (D_i, D_j) = \sum_{k=1}^{r+1} c_{ijk} D_k \ddagger \quad (i, j = 1, 2, \cdots, r+1).$$

The group Γ has $r+1$ essential parameters (i.e., there is no linear relation with constant coefficients between $D_1, \cdots, D_{r+1}$) if, and only if, G contains no exceptional§ infinitesimal transformation, i.e., no infinitesimal transformation $\sum_{i=1}^{r+1} a_i X_i$ such that

$$(4) \qquad (\sum_{i=1}^{r+1} a_i X_i, X_j) = \sum_{i=1}^{r+1} a_i (X_i, X_j) = 0 \quad (j = 1, 2, \cdots, r+1).$$

Corresponding to each independent exceptional infinitesimal transformation of G, $\sum a_i X_i$, is an independent linear relation $\sum_{i=1}^{r+1} a_i D_i = 0$ between the differential operators $D_1, \cdots, D_r, D_{r+1}$.||

We shall assume in what follows that the group G has just one exceptional infinitesimal transformation, which we may take without loss of generality to be X_{r+1}. In this case,

$$(5) \qquad \begin{aligned} (X_i, X_j) &= \sum_{k=1}^{r} c_{ijk} X_k + c_{i,\, j,\, r+1} X_{r+1} \quad &(i, j = 1, 2, \cdots, r), \\ (X_i, X_{r+1}) &= (X_{r+1}, X_i) = 0 &(i = 1, 2, \cdots, r, r+1); \end{aligned}$$

* Lie-Scheffers, Continuirliche Gruppen, p. 391.

† Lie, loc. cit., p. 466.

‡ Lie, loc. cit., p. 467.

§ Lie calls it "ausgezeichnete," loc. cit., p. 465.

|| Lie, loc. cit., p. 465.

so that

(6) $$c_{i,\,r+1,\,k} = -\,c_{r+1,\,i,\,k} = 0 \quad (i, k = 1, 2, \cdots, r, r+1);$$

and, therefore,

(7) $$D_{r+1} = 0, \qquad D_i = \sum_{j=1}^{r}\sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k},$$
$$(D_i, D_j) = \sum_{k=1}^{r} c_{ijk} D_k \quad (i, j = 1, \cdots, r).$$

In consequence of the assumption that G contains just one exceptional transformation, there will exist a finite continuous group G' with r essential parameters generated by r infinitesimal transformations, whose symbols we shall denote by $Y_1, \cdots, Y_r$, such that

(8) $$(Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k\text{*} \qquad (i, j = 1, 2, \cdots, r).$$

Let Γ' denote the adjoint of G', and the symbols of its infinitesimal transformations let be $D'_1, \cdots, D'_r$, where

(9) $$D'_i \equiv \sum\sum \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r).$$

We have then

(10) $$(D'_i, D'_j) = \sum_{k=1}^{r} c_{ijk} D'_k \qquad (i, j = 1, 2, \cdots, r).$$

In what follows we shall investigate the conditions that may be imposed on the structure of G' (when, as assumed, G has just one exceptional infinitesimal transformation) so that we shall have

(11) $$c_{i,\,k,\,r+1} = -\,c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1);$$

and we shall show, if there is just one spread invariant to the adjoint of G' (which will be assumed in what follows), that then

(11) $$c_{i,\,k,\,r+1} = -\,c_{k,\,i,\,r+1} = 0 \quad (i, k = 1, \cdots, r, r+1),$$

or, by a suitable choice of the X's, these conditions will be satisfied. From what is stated below, it follows that, when the adjoint of G' has but one invariant spread, this spread is an $(r-1)$-spread.

Invariant spreads of the adjoint.—Lie shows† that the invariants of the adjoint of any finite continuous group may be taken homogeneous: if there is only one invariant, it will be homogeneous but not of order zero; if there are two or more invariants, all of them can be taken homogeneous and of order zero except one, which will be homogeneous but not of order zero.

* Lie-Engel, "Transformations Gruppen," Vol. I, pp. 300–305.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 596–599.

Every function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint Γ' of G', therefore, yields a spread invariant to Γ'. Since, by supposition, the adjoint of G' has but one invariant spread, there is therefore but one function of $\alpha_1, \cdots, \alpha_r$ invariant to the adjoint of G'. This function will be denoted, in this paper, by $\varphi \equiv \varphi(\alpha_1, \cdots, \alpha_r)$, homogeneous in the α's; and then $\varphi(\alpha) = 0$ will be the only spread invariant to the adjoint of G'.

The function $\varphi(\alpha)$ is a solution of the system of equations

$$D'_i f(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial f(\alpha)}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial f(\alpha)}{\partial \alpha_r} = 0 \quad (i = 1, 2, \cdots, r). \tag{12}$$

From $\partial\varphi(\alpha)/\partial\alpha_{r+1} = 0$ it follows, by (7), that

$$D_i\varphi(\alpha) \equiv \sum_{j=1}^{r} \alpha_j c_{ji1} \frac{\partial \varphi(\alpha)}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{jir} \frac{\partial \varphi(\alpha)}{\partial \alpha_r} \equiv D'_i\varphi(\alpha) = 0 \quad (i = 1, 2, \cdots, r); \tag{13}$$

therefore, $\varphi(\alpha)$ is also an invariant of the adjoint of G.

It is convenient to denote by

$$E_i \equiv \begin{pmatrix} c_{i11}, & \cdots, & c_{ir1}, & 0 \\ c_{i12}, & \cdots, & c_{ir2}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr}, & 0 \\ c_{i,1,r+1}, & \cdots, & c_{i,r,r+1}, & 0 \end{pmatrix} \quad (i = 1, 2, \cdots, r) \tag{14}$$

the matrix of the differential operators

$$D_i \equiv \sum_{j=1}^{r} \sum_{k=1}^{r+1} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r).$$

Then

$$\sum a_i E_i \equiv \begin{pmatrix} \sum a_i c_{i,1,1}, & \cdots, & \sum a_i c_{i,r,1}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sum a_i c_{i,1,r}, & \cdots, & \sum a_i c_{i,r,r}, & 0 \\ \sum a_i c_{i,1,r+1}, & \cdots, & \sum a_i c_{i,r,r+1}, & 0 \end{pmatrix} \tag{15}$$

is the matrix of the general infinitesimal transformation $\sum_{i=1}^{r} a_i D_i$ of Γ. Similarly we shall write

$$\mathcal{E}_i \equiv \begin{pmatrix} c_{i11}, & \cdots, & c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{i1r}, & \cdots, & c_{irr} \end{pmatrix} \qquad (i = 1, 2, \cdots, r) \tag{16}$$

to denote the matrix of the differential operators

$$D'_i \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r);$$

and then

$$(17) \qquad \sum_1^r a_i \mathcal{E}_i \equiv \begin{vmatrix} \sum a_i c_{i11}, & \cdots, & \sum a_i c_{ir1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \sum a_i c_{i1r}, & \cdots, & \sum a_i c_{irr} \end{vmatrix}$$

will denote the matrix of the general infinitesimal transformation $\sum_{i=1}^r a_i D'_i$ of Γ'.

Any invariant of the adjoint of G being a solution of the complete system of equations

$$(18) \qquad D_i f(\alpha) \equiv \sum_{j=1}^{r+1}\sum_{k=1} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r),$$

the number of invariants of the group Γ, adjoint to G, is determined by the array

$$(19) \qquad \begin{matrix} \sum_1^r \alpha_j c_{j11}, & \cdots, & \sum_1^r \alpha_j c_{j1r}, & \sum \alpha_j c_{j,\,1,\,r+1} \\ \cdot & \cdot & \cdot & \cdot \\ \sum \alpha_j c_{jr1}, & \cdots, & \sum \alpha_j c_{jrr}, & \sum \alpha_j c_{j,\,r,\,r+1} \end{matrix}$$

being equal to $r + 1$ less the order of the non-zero determinant of highest order formed from the array, and thus to the nullity of $\sum \alpha_j \breve{E}_j$, where $\breve{E}_j$ denotes the transverse of E_j. Therefore, the number of invariants of G is equal to the nullity of $\sum \alpha_j E_j$, being a solution of the complete system of equations

$$(20) \qquad D'_i f(\alpha) \equiv \sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial f(\alpha)}{\partial \alpha_k} = 0 \qquad (i = 1, 2, \cdots, r).$$

The number of invariants of the adjoint of G' is equal to the nullity of the array

$$(21) \qquad \begin{matrix} \sum_1^r \alpha_j c_{j11}, & \cdots, & \sum_1^r \alpha_j c_{j1r} \\ \cdot & \cdot & \cdot \\ \sum_1^r \alpha_j c_{jr1}, & \cdots, & \sum_1^r \alpha_j c_{jrr} \end{matrix}$$

and thus is equal to the nullity of $\sum_1^r \alpha_j \breve{\mathcal{E}}_j$ (where, as before, $\breve{\mathcal{E}}_j$ is the transverse of $\mathcal{E}_j$), or, what is the same thing, to the nullity of $\sum_1^r \alpha_j \mathcal{E}_j$.

Since it is assumed that the adjoint of G' has just one invariant, it follows that the nullity of $\sum_1^r \alpha_i \mathcal{E}_i$ is one for an arbitrary system of values

of the α's, i.e., at least one of the minors of $|\sum\alpha_i\mathcal{E}_i|$* of order $r-1$ is not zero. But every minor of $|\sum\alpha_i\mathcal{E}_i|$ is a minor of $|\sum\alpha_iE_i|$, therefore at least one minor of the order $r-1$ of the determinant $|\sum\alpha_iE_i|$ is not zero, and thus the nullity of the matrix $\sum\alpha_iE_i$ *cannot exceed two* for an arbitrary system of values of the α's.

Further, for $\alpha_1, \cdots, \alpha_r, \alpha_{r+1}$ assigned, the symbolic equation

$$\left(\sum_1^{r+1}\alpha_iX_i,\ \sum_1^{r+1}\eta_iX_i\right) = 0, \tag{22}$$

is satisfied for

$\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = 0$ and $\eta_1 = 0, \cdots, \eta_r = 0, \eta_{r+1} = 1$.

But from (22) follows

$$\sum\alpha_iE_i(\eta_1, \cdots, \eta_{r+1}) = \begin{pmatrix} \sum\alpha_ic_{i11}, & \cdots \\ \cdot & \cdot \\ \sum\alpha_ic_{i,1,r+1}, & \cdots \end{pmatrix}(\eta_1, \cdots, \eta_{r+1}) = 0.\dagger \tag{23}$$

Therefore, this system of linear equations has two independent solutions, namely,

$\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = 0$ and $\eta_1 = 0, \cdots, \eta_r = 0, \eta_{r+1} = 1$.

Whence it follows that the nullity of the matrix $\sum\alpha_iE_i$, for any system of values of the α's, is at least two. Wherefore, the nullity of $\sum\alpha_iE_i$, for an arbitrary system of values of the α's, is just *equal to two;* and thus the number of *invariants* of the adjoint of G is *equal to two.*

The Holomorphic Invariant of the Adjoint.—Let the second invariant of the adjoint of G, independent of $\varphi(\alpha)$, be $\psi \equiv \psi(\alpha, \cdots, \alpha_{r+1})$ homogeneous in the α's. We must have

$$\frac{\partial\psi(\alpha)}{\partial\alpha_{r+1}} \not\equiv 0. \tag{24}$$

For, otherwise, if $\dfrac{\partial\psi(\alpha)}{\partial\alpha_{r+1}} \equiv 0$, we get

$$D_i'\psi(\alpha) \equiv D_i\psi(\alpha) \equiv 0, \tag{25}$$

and then, since the group adjoint to G' has but one invariant, we should have $\psi = W(\varphi)$, which is contrary to our assumption.

By Lie's theorem, as stated above, $\psi(\alpha)$ can be taken homogeneous (and of order zero, since a homogeneous invariant of the adjoint of G, viz., $\varphi(\alpha_1, \cdots, \alpha_r)$, not of order zero, exists); and thus, we may put

$$\psi(\alpha) \equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}}\right). \tag{26}$$

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* The determinant of any matrix M will be denoted by $|M|$.

† Lie-Scheffers, "Continuirliche Gruppen," pp. 558, 562. The notation used here is due to Cayley, *Philosophical Transactions*, 1858.

Let

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \cdots, \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \beta_{r+1} = \alpha_{r+1}; \tag{27}$$

then

$$\alpha_1 = \beta_1 \beta_{r+1}, \cdots, \alpha_r = \beta_r\beta_{r+1}, \alpha_{r+1} = \beta_{r+1} \tag{28}$$

and

$$\frac{\partial F(\beta)}{\partial \beta_{r+1}} \equiv 0. \tag{29}$$

Let

$$\begin{aligned}\bar{D}_i &\equiv D_i\beta_1 \frac{\partial}{\partial\beta_1} + \cdots + D_i\beta_r \frac{\partial}{\partial\beta_r} + D_i\beta_{r+1}\frac{\partial}{\partial\beta_{r+1}} \\ &\equiv \left(\sum_{j=1}^{r} c_{ji1}\beta_j - \beta_1 \sum_{j=1}^{r} c_{jir+1}\right)\frac{\partial}{\partial\beta_1} \\ &\quad + \cdots + \left(\sum_{j=1}^{r} c_{jir}\beta_j - \beta_r \sum_{1}^{r} c_{jir+1}\beta_j\right)\frac{\partial}{\partial\beta_r} \\ &\quad + \beta_{r+1}\sum_{1}^{r} c_{jir+1}\beta_j \frac{\partial}{\partial\beta_{r+1}} \quad (i = 1, 2, \cdots, r).\end{aligned} \tag{30}$$

Then, if $f(\alpha) \equiv \mathfrak{f}(\beta)$,

$$\bar{D}_i\mathfrak{f}(\beta) = D_i f(\alpha) \qquad (i = 1, 2, \cdots, r). \tag{31}$$

Therefore,

$$\begin{aligned}\left(\sum_{1}^{r} c_{ji1}\beta_j - \beta_1\sum_{1}^{r} c_{jir+1}\beta_j\right)\frac{\partial F(\beta)}{\partial\beta_1} + \cdots + \beta_{r+1}\sum_{1}^{r} c_{jir+1}\beta_j\frac{\partial F(\beta)}{\partial\beta_{r+1}} \\ \equiv \bar{D}_iF(\beta) \equiv D_i\psi(\alpha) \equiv 0 \quad (i = 1, 2, \cdots, r).\end{aligned} \tag{32}$$

It follows that $F(\beta)$ is a solution of the system of equations

$$\bar{D}_1 f(\beta) = 0, \cdots, \bar{D}_r f(\beta) = 0. \tag{33}$$

Since the coefficients of this system of equations are rational integral functions of the β's, the solution $F(\beta)$ of this system may be taken holomorphic in the neighborhood of $\beta_1 = 0, \cdots, \beta_r = 0, \beta_{r+1} = 1$. Consequently

$$\psi(\alpha) \equiv F\left(\frac{\alpha_1}{\alpha_{r+1}}, \cdots, \frac{\alpha_r}{\alpha_{r+1}}\right)$$

may be taken holomorphic in the neighborhood of $\alpha_1 = 0, \cdots, \alpha_r = 0, \alpha_{r+1} = 1$. There are two cases: first, $\psi(\alpha)$ is algebraic; second, $\psi(\alpha)$ is transcendental.

The Function ψ (α) Algebraic Invariant.—The invariant $\psi(\alpha)$ is now assumed to be algebraic. In this case the invariant spread $\psi(\alpha) = 0$ is represented by the rational integral equation of degree m homogeneous in the α's,

$$\Psi(\alpha) \equiv u_0(\alpha)\alpha_{r+1}^{m} + u_1(\alpha)\alpha_{r+1}^{m-1} + \cdots + u_k(\alpha)\alpha_{r+1}^{m-k} + \cdots + u_m(\alpha) = 0, \tag{34}$$

where $u_k(\alpha)$ is a polynomial of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$.

Since the point $(0, \cdots, 0, 1)$, i.e., the point $\alpha_1 = \cdots = \alpha_r = 0, \alpha_{r+1} = 1$, corresponding to the exceptional infinitesimal transformation X_{r+1} of G, is invariant to the adjoint of G,* the polars of $(0, \cdots, 0, 1)$ quâ $\Psi(\alpha) = 0$ are all invariant to the adjoint of G. These polars are:

$$
\begin{aligned}
\frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} &\equiv \frac{m!}{(m-1)!}u_0(\alpha)\alpha_{r+1}^{m-1} + \frac{(m-1)!}{(m-2)!}u_1(\alpha)\alpha_{r+1}^{m-2} \\
&\qquad\qquad + \cdots + \frac{1!}{0!}u_{m-1}(\alpha) = 0, \\
\frac{\partial^2\Psi(\alpha)}{\partial\alpha_{r+1}^2} &\equiv \frac{m!}{(m-2)!}u_0(\alpha)\alpha_{r+1}^{m-2} + \frac{(m-1)!}{(m-3)!}u_1(\alpha)\alpha_{r+1}^{m-3} \\
&\qquad\qquad + \cdots + \frac{2!}{0!}u_{m-2}(\alpha) = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial^{m-2}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-2}} &\equiv \frac{m!}{2!}u_0(\alpha)\alpha_{r+1}^2 + \frac{(m-1)!}{1!}u_1(\alpha)\alpha_{r+1} \\
&\qquad\qquad + \frac{(m-2)!}{0!}u_2(\alpha) = 0, \\
\frac{\partial^{m-1}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-1}} &\equiv \frac{m!}{1!}u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!}u_1(\alpha) = 0.
\end{aligned} \tag{35}
$$

In the equation

$$\Psi(\alpha) \equiv u_0(\alpha)\alpha_{r+1}^m + u_1(\alpha)\alpha_{r+1}^{m-1} + \cdots + u_m(\alpha) = 0$$

we may either have $u_0(\alpha) \neq 0$, in which case the point $(0, \cdots, 0, 1)$ is not on the spread $\Psi(\alpha) = 0$, or $u_0(\alpha) = 0$, and thus the point $(0, \cdots, 0, 1)$ is on the spread $\Psi(\alpha) = 0$. In the former case, there is an $(r-1)$-flat not passing through the point $(0, \cdots, 0, 1)$, namely, the $(m-1)$th polar of that point quâ $\Psi(\alpha) = 0$,

$$\frac{\partial^{m-1}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-1}} \equiv \frac{m!}{1!}u_0(\alpha)\alpha_{r+1} + \frac{(m-1)!}{0!}u_1(\alpha) = 0, \tag{36}$$

invariant to the adjoint of G.

Let $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(r)}$, where

$$\alpha^{(\nu)} \equiv (\alpha_1^{(\nu)}, \cdots, \alpha_r^{(\nu)}, \alpha_{r+1}^{\nu}) \qquad (\nu = 1, \cdots, r),$$

be any r points in the $(r-1)$-flat $\dfrac{\partial^{m-1}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-1}} = 0$ not lying in any $(r-2)$-flat. These r points together with the point $\alpha^{(r+1)} \equiv (0, \cdots, 0, 1)$ then constitute a system of $r+1$ points not lying in any r-flat.

Let us transform the coördinate system by the transformation

$$\alpha_i = \bar{\alpha}_1\alpha_i^{(1)} + \cdots + \bar{\alpha}_r\alpha_i^{(r)} + \bar{\alpha}_{r+1}\alpha_i^{(r+1)} \quad (i = 1, 2, \cdots, r, r+1), \tag{37}$$

* Lie-Scheffers, "Continuirliche Gruppen," pp. 465, 485.

where

$$\alpha_i^{(r+1)} = 0 \quad (i \neq r+1), \qquad \alpha_{r+1}^{(r+1)} = 1.$$

In Cayley's notation this transformation can be written:

$$(\alpha_1, \cdots, \alpha_r, \alpha_{r+1}) = \begin{pmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_1^{(r)}, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{r+1}^{(1)}, & \cdots, & \alpha_{r+1}^{(r)}, & 0 \end{pmatrix} (\bar{\alpha}_1, \cdots, \bar{\alpha}_r, \bar{\alpha}_{r+1}).$$

And now if

$$(38) \qquad \bar{X}_i = \alpha_i^{(1)} X_1 + \cdots + \alpha_r^{(1)} X_r + \alpha_{r+1}^{(i)} X_{r+1} \quad (i = 1, 2, \cdots, r, r+1),$$

or

$$(\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}) = \begin{pmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{pmatrix} (X_1, \cdots, X_r, X_{r+1}),$$

then

$$(39) \qquad \sum_{i=1}^{r+1} \bar{\alpha}_i \bar{X}_i = \sum_{i=1}^{r+1} \alpha_i X_i.$$

The infinitesimal transformations $\bar{X}_1, \cdots, \bar{X}_r, \bar{X}_{r+1}$ will then be independent, since the determinant

$$\begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)}, & \alpha_{r+1}^{(1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)}, & \alpha_{r+1}^{(r)} \\ 0, & \cdots, & 0, & 1 \end{vmatrix} = \begin{vmatrix} \alpha_1^{(1)}, & \cdots, & \alpha_r^{(1)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_1^{(r)}, & \cdots, & \alpha_r^{(r)} \end{vmatrix} \neq 0;$$

and we shall have

$$(40) \qquad (\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r+1} \bar{c}_{ijk} \bar{X}_k \quad (i, j = 1, 2, \cdots, r, r+1).$$

Since $\bar{X}_1, \cdots, \bar{X}_r$ are represented by the points $\alpha^{(1)}, \cdots, \alpha^{(r)}$ in the $(r-1)$-flat

$$\frac{(m-1)!}{1!} u_0(\alpha) \alpha_{r+1} + \frac{(m-1)!}{0!} u_1(\alpha) = 0$$

which is invariant to the adjoint of G, they will constitute an invariant subgroup of G;* and thus

$$(41) \qquad (\bar{X}_i, \bar{X}_j) = \sum_{k=1}^{r} c_{ijk} \bar{X}_k \quad (i = 1, 2, \cdots, r; j = 1, \cdots, r, r+1).$$

* Lie-Scheffers, "Continuirliche Gruppen," pp. 485–487.

Further, $\bar{X}_{r+1} = X_{r+1}$. Whence

$$\bar{c}_{ijr+1} = -\bar{c}_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1), \tag{42}$$

which was to be proved. We may therefore assume that $u_0(\alpha) = 0$.

If now $m = 1$, the equation of the spread invariant to the adjoint of G is $\Psi(\alpha) \equiv u_1(\alpha) = 0$. But in this case $\dfrac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv 0$, which is contrary to our assumption.

Let $m = 2$. The equations of the invariant spread and the invariant polars are then

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0, \qquad \frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv u_1(\alpha) = 0.$$

Let

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv 0, \qquad u_2(\alpha) \not\equiv u_1(\alpha)\cdot v_1(\alpha),$$

where $v_1(\alpha)$ is a function linear in $\alpha_1, \cdots, \alpha_r$. If now

$$u_1(\alpha) = 0, \qquad \text{then} \qquad \frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} = 0;$$

and since this equation is invariant, $u_1(\alpha') = \dfrac{\partial\Psi(\alpha')}{\partial\alpha'_{r+1}} = 0$, where $\alpha'_1, \cdots, \alpha'_r$, α'_{r+1} are the coördinates of the point obtained by applying the infinitesimal transformations of the adjoint of G to the point $(\alpha_1, \cdots, \alpha_{r+1})$. Again, if

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0,$$

then

$$\frac{\partial\Psi(\alpha)}{\partial\alpha_{r+1}} \equiv u_1(\alpha) = 0, \qquad \Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} + u_2(\alpha) = 0;$$

and, since these equations are separately invariant to the adjoint of G, it follows that

$$\frac{\partial\Psi(\alpha')}{\partial\alpha'_{r+1}} \equiv u_1(\alpha') = 0, \qquad \Psi(\alpha') \equiv u_1(\alpha')\alpha'_{r+1} + u_2(\alpha') = 0;$$

and thus

$$u_1(\alpha') = 0, \qquad u_2(\alpha') = 0.$$

Therefore, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to the assumption that the adjoint of G' has but one invariant spread.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv 0,$$

then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1} = 0$$

is reducible. In this case each component of that spread is invariant to the adjoint of G; and therefore,

$$\alpha_{r+1} = 0$$

is an invariant flat not passing through the point $(0, \cdots, 0, 1)$, a case already treated above. In this case we can clearly see that

$$c_{ijr+1} = -c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1).$$

If

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0,$$

then

$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv 0,$$

which is contrary to the assumption.

If

$$u_1(\alpha) \not\equiv 0, \qquad u_2(\alpha) \equiv v_1(\alpha) \cdot u_1(\alpha) \not\equiv 0,$$

where $v_1(\alpha)$ denotes a function linear in $\alpha_1, \cdots, \alpha_r$, then the invariant spread

$$\Psi(\alpha) \equiv u_1(\alpha) \cdot \alpha_{r+1} + u_2(\alpha) \equiv u_1(\alpha)[\alpha_{r+1} + v_1(\alpha)] = 0$$

is reducible; and the component flat

$$\alpha_{r+1} + v_1(\alpha) = 0$$

is invariant and does not pass through the point $(0, \cdots, 0, 1)$, in which case, as was shown above, we can, by a suitable choice of $X_1, \cdots, X_r, X_{r+1}$, make

$$c_{ijr+1} = -c_{jir+1} = 0 \quad (i, j = 1, 2, \cdots, r, r+1).$$

Let now m be any positive integer greater than 2. The equations of the invariant spread and the invariant polars are then

$$\Psi(\alpha) \equiv u_1(\alpha)\alpha_{r+1}^{m-1} + u_2(\alpha)\alpha_{r+1}^{m-2} + \cdots + u_m(\alpha) = 0,$$
$$\frac{\partial \Psi(\alpha)}{\partial \alpha_{r+1}} \equiv \frac{(m-1)!}{(m-2)!} u_1(\alpha) \cdot \alpha_{r+1}^{m-2} + \cdots + \frac{1!}{0!} u_{m-1}(\alpha) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\frac{\partial^{m-2} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-2}} \equiv \frac{(m-1)!}{1!} u_1(\alpha)\alpha_{r+1} + \frac{(m-2)!}{0!} u_2(\alpha) = 0,$$
$$\frac{\partial^{m-1} \Psi(\alpha)}{\partial \alpha_{r+1}^{m-1}} \equiv \frac{(m-1)!}{0!} u_1(\alpha) = 0.$$

First, let $u_1(\alpha) \not\equiv 0$. Then either $u_2(\alpha) \not\equiv 0$ does not contain $u_1(\alpha)$, or

$u_2(\alpha)$, $u_3(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_1(\alpha)$. In the former case, as we have seen before, the spread

$$u_1(\alpha) = 0$$

and the spread

$$u_1(\alpha) = 0, \qquad u_2(\alpha) = 0$$

are both invariant to the adjoint of G', which is contrary to our assumption. In the latter case

$$\begin{aligned} u_2(\alpha) &= v_1(\alpha) \cdot u_1(\alpha), \\ u_3(\alpha) &= v_2(\alpha) \cdot u_1(\alpha), \\ &\cdots\cdots \\ u_{p-1}(\alpha) &= v_{p-2}(\alpha) \cdot u_1(\alpha), \\ u_p(\alpha) &= v_{p-1}(\alpha) \cdot u_1(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of degree k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial \alpha_{r+1}^{m-q}} = 0 \qquad (q = 2, 3, \cdots, p)$$

is reducible. Therefore, the component flat

$$\frac{(m-1)!}{1!}\alpha_{r+1} + \frac{(m-2)!}{0!} v_1(\alpha) = 0$$

is invariant. This case has been treated above.

Secondly, let

$$u_1(\alpha) \equiv 0, \qquad u_2(\alpha) \not\equiv 0.$$

Either $u_3(\alpha) \not\equiv 0$ does not contain $u_2(\alpha)$, or $u_3(\alpha)$, $u_4(\alpha)$, $\cdots$, $u_p(\alpha)$ $(p \leqq m)$ each contain $u_2(\alpha)$. In the former case, by reasoning similar to that employed before, it appears that the spread

$$u_2(\alpha) = 0$$

and the spread

$$u_2(\alpha) = 0, \qquad u_3(\alpha) = 0$$

are both invariant to the adjoint of G'. In the latter case

$$\begin{aligned} u_3(\alpha) &= v_1(\alpha) \cdot u_2(\alpha), \\ &\cdots\cdots \\ u_{p-1}(\alpha) &= v_{p-3}(\alpha) \cdot u_2(\alpha), \\ u_p(\alpha) &= v_{p-2}(\alpha) \cdot u_2(\alpha), \end{aligned}$$

where $v_k(\alpha)$ is a function of order k homogeneous in $\alpha_1, \cdots, \alpha_r$. In this case each of the invariant spreads

$$\frac{\partial^{m-q}\Psi(\alpha)}{\partial \alpha_{r+1}^{m-q}} = 0 \qquad (q = 3, 4, \cdots, p)$$

is reducible. In particular the invariant spread

$$\frac{\partial^{m-3}\Psi(\alpha)}{\partial\alpha_{r+1}^{m-3}} \equiv \frac{(m-2)!}{2!} u_2(\alpha)\cdot\alpha_{r+1} + \frac{(m-3)!}{1!} u_3(\alpha)$$
$$\equiv u_2(\alpha)\left[\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!} v_1(\alpha)\right] = 0$$

is reducible. Therefore, the component flat

$$\frac{(m-2)!}{2!}\alpha_{r+1} + \frac{(m-3)!}{1!} v_1(\alpha) = 0$$

is invariant. This case has also been treated above.

By the same reasoning we may treat the other similar cases which arise.

The Function $\psi(\alpha)$ Transcendental Invariant.—Since the point $\alpha^{(1)} = (0, \cdots, 0, 1)$ is invariant to the adjoint of G, the spreads

$$\sum_{i=1}^{r+1}\alpha_i\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\sum_{i=1}^{r+1}\sum_{j=1}^{r+1}\alpha_i\alpha_j\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\sum_{i_1=1}^{r+1}\cdots\sum_{i_\mu=1}^{r+1}\alpha_{i_1}\cdots\alpha_{i_\mu}\left[\frac{\partial^\mu\psi(\alpha)}{\partial\alpha_{i_1}\cdots\partial\alpha_{i_\mu}}\right]_{\alpha=\alpha^{(1)}} = 0,$$

etc. $\cdots\cdots\cdots\cdots$

are invariant to the adjoint of G. Wherefore, there is an algebraic spread invariant to the adjoint of G, a case already treated above, or all the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are zero for $\alpha = \alpha^{(1)}$. But this is impossible as is shown below.

We have seen that

$$\psi(\alpha) = F(\beta_1, \cdots, \beta_r),$$

where

$$\beta_1 = \frac{\alpha_1}{\alpha_{r+1}}, \qquad \cdots, \qquad \beta_r = \frac{\alpha_r}{\alpha_{r+1}}, \qquad \beta_{r+1} = \alpha_{r+1},$$

can be taken analytic in the neighborhood of the point $\beta_1^{(1)} = \cdots = \beta_r^{(1)} = 0$, $\beta_{r+1}^{(1)} = 1$; and, therefore,

$$\psi(\alpha) = F(\beta) = F(\beta^{(1)}) + \sum_{i=1}^{r}\beta_i\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}}$$
$$+ \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\beta_i\beta_j\left[\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j}\right] + \cdots + \cdots.$$

Since

$$\frac{\partial\psi(\alpha)}{\partial\alpha_i} = \frac{\partial F(\beta)}{\partial\beta_i}\cdot\frac{1}{\alpha_{r+1}} \qquad (i = 1, 2, \cdots, r),$$
$$\frac{\partial\psi(\alpha)}{\partial\alpha_{r+1}} = -\frac{\alpha_1}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_1} - \cdots - \frac{\alpha_r}{\alpha_{r+1}^2}\cdot\frac{\partial F(\beta)}{\partial\beta_r},$$

we should have, if

$$\left[\frac{\partial\psi(\alpha)}{\partial\alpha_i}\right]_{\alpha=\alpha^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r, r+1),$$

$$\left[\frac{\partial F(\beta)}{\partial\beta_i}\right]_{\beta=\beta^{(1)}} = 0 \qquad (i = 1, 2, \cdots, r).$$

Since

$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j} \qquad (i = 1, 2, \cdots, r),$$

$$\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_{r+1}} = \frac{\partial^2\psi(\alpha)}{\partial\alpha_{r+1}\partial\alpha_i} = \frac{1}{\alpha_{r+1}^2}\cdot\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_{r+1}} \qquad (i = 1, 2, \cdots, r+1),$$

we should have, if

$$\left[\frac{\partial^2\psi(\alpha)}{\partial\alpha_i\partial\alpha_j}\right]_{\alpha=\alpha^{(1)}} = 0 \quad (i = 1, 2, \cdots, r+1),$$

$$\left[\frac{\partial^2 F(\beta)}{\partial\beta_i\partial\beta_j}\right]_{\beta=\beta^{(1)}} = 0 \quad (i = 1, 2, \cdots, r), \text{ etc. } \cdots.$$

Thus in case the partial differential coefficients of $\psi(\alpha)$ quâ $\alpha_1, \cdots, \alpha_{r+1}$ are all zero for $\alpha = \alpha^{(1)}$, then

$$\psi(\alpha) \equiv F(\beta) \equiv 0,$$

which is contrary to the assumption.

Note.—Not in all the groups with $r + 1$ essential parameters and one exceptional infinitesimal transformations can all the c_{ijr+1}'s $(i, j = 1, 2, \cdots, r)$ be made zero. The group with 5 essential parameters and one exceptional infinitesimal transformation, isomorphic (not holoedrically) with the integrable group of 4 essential parameters of the type,

$$(Y_1, Y_2) = 0, \qquad (Y_1, Y_3) = 0, \qquad (Y_2, Y_3) = Y_1,$$
$$(Y_1, Y_4) = Y_1, \qquad (Y_2, Y_4) = Y_2, \qquad (Y_3, Y_4) = 0,\text{*}$$

may serve as an illustration where not all the c_{ijr}'s $(i, j = 1, 2, 3, 4)$ can be made zero.

CLARK UNIVERSITY,
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* Lie-Scheffers, "Continuirliche Gruppen," p. 582.